

# Linear Algebra I

26/01/2015, Monday, 9:00-12:00

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You are **NOT** allowed to use any type of calculators.

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**1** (8 + 7 = 15 pts)

**Linear equations**

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Consider the system of equations

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 4 \\ b \end{bmatrix}.$$

- (a) Determine all values of  $a$  and  $b$  such that this system is
- (i) inconsistent.
  - (ii) consistent.
- (b) Find the set of solutions for
- (i)  $a = b = 1$ .
  - (ii)  $a = b = 2$ .

**2** (15 pts)

**Determinants**

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Find the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & p & 0 & q \\ r & 2 & s & 1 & t \\ 0 & 0 & 1 & 0 & 0 \\ u & 1 & v & 1 & w \\ 0 & 0 & x & 0 & 2 \end{bmatrix}.$$

**3** (2 + 5 + 8 = 15 pts)

**Vector spaces**

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Consider the vector space  $P_3$  of polynomial with degree less than 3.

- (a) Are the vectors  $\{x^2 + 2x + 3, x^2 + x + 2, x^2 - x\}$  linearly independent?
- (b) Is the set

$$\{p \in P_3 \mid p(x) + x^2 p\left(\frac{1}{x}\right) = 0\}$$

a subspace of  $P_3$ ? If so, find its dimension.

- (c) Let  $T$  be a transformation from  $P_3$  to  $P_3$  given by

$$T(p(x)) := p(x) + x^2 p\left(\frac{1}{x}\right).$$

Is  $T$  a linear transformation? If so, find its matrix representation with respect to the basis  $\{1, x, 1 + 2x + x^2\}$ .

4 (15 pts)

Least squares problem

Let  $A$  be an  $m \times n$  matrix with  $\text{rank}(A) = n$ . Suppose that the columns of  $A$  are orthogonal. Show that the least squares solution of the system  $Ax = b$  is given by

$$x = \begin{bmatrix} \frac{b^T a_1}{a_1^T a_1} \\ \frac{b^T a_2}{a_2^T a_2} \\ \vdots \\ \frac{b^T a_n}{a_n^T a_n} \end{bmatrix}$$

where  $a_i$  denotes the  $i$ -th column of  $A$ .

5 (1 + 3 + 2 + 3 + 3 + 3 = 15 pts)

Eigenvalues

Let  $M$  be a  $3 \times 3$  matrix with the characteristic polynomial  $p_M(\lambda) = \lambda^3 - \lambda$ .

- Find the determinant of  $M$ . Is it nonsingular?
- Find the characteristic polynomial of  $M^2$ .
- Is  $M$  diagonalizable?
- Is  $M^2$  diagonalizable?
- Let  $T$  be a nonsingular matrix. Find the eigenvalues of  $TMT^{-1}$ . Is it diagonalizable?
- Find the eigenvalues of  $M - I_3$  where  $I_3$  denotes the  $3 \times 3$  identity matrix.

6 (5 + 10 = 15 pts)

Eigenvalues and partitioned matrices

Let  $A$  be an  $n \times n$  nonsingular matrix and  $I_n$  denote the  $n \times n$  identity matrix. Also let  $M$  be the  $2n \times 2n$  matrix given by

$$M = \begin{bmatrix} I_n & A \\ A^{-1} & I_n \end{bmatrix}.$$

- Show that if  $\lambda$  is an eigenvalue of  $M$  then either  $\lambda = 0$  or  $\lambda = 2$ .
- Is  $M$  diagonalizable? If so, find a diagonalizer for  $M$ .

10 pts free